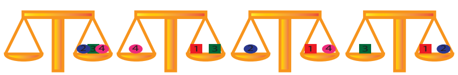
NEGACYCLIC WEIGHING MATRICES

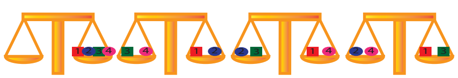
Basic Notions

A **Weighing Matrix** W of order n and weight k is a $n \times n$ matrix with k nonzero entries, 1 or $-$ ($-$ stands for -1) in each row, and $WW^t = kI_n$, where I_n is the identity matrix of order n. Such a matrix gives instructions for weighing n objects in n weighings with a 2-pan scale. Each row is a weighing, the *ith* object is placed in the right pan if the *ith* entry is 1, in the left pan for $-$, and omitted from the weighing if the entry is 0. When these instructions are followed, this will guarantee the measured weight in each object will have the smallest possible error. We write $W = W(n,k)$ where k is called the *weight* and n is called the *order*. $W(n, n-1)$ is called a **Conference Matrix**, and $W(n,n)$ is called a **Hadamard Matrix** (denoted by H_n).

Conference Matrix $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ 1 & 1 & - & 0 \end{bmatrix}$

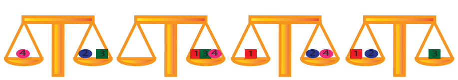


Hadamard Matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & - & 1 \\ 1 & - & 1 & - \end{bmatrix}$



A matrix M is called a **Circulant Matrix** if each row except the first is a right cyclic shift by one position of the previous row. Such a matrix is denoted by circ($a_1 a_2 ... a_n$), where ($a_1 a_2 ... a_n$) is the first row of the matrix. M is called a **Negacyclic Matrix** if M has all the same properties described above for circ($a_1 a_2 ... a_n$), except for each row ($b_1 b_2 ... b_n$), $b_1 = -c_n$, where c_n is the last entry of the row preceding it. Such a matrix is denoted NC($a_1 a_2 ... a_n$), where ($a_1 a_2 ... a_n$) is the first row of the matrix. If M is also a weighing matrix, M is called a **Negacyclic Weighing Matrix**, denoted NCW(n,k). Below is an example of a NCW(4,3)

Negacyclic Weighing Matrix $\begin{bmatrix} 0 & 1 & 1 & - \\ 1 & 0 & 1 & 1 \\ - & 1 & 0 & 1 \\ - & - & 1 & 0 \end{bmatrix}$



Weighing matrices have many applications. And so the question *Given order n and weight k, does W(n,k) exist?* is what drives the study of such matrices. Since Negacyclic weighing matrices is a subclass of weighing matrices, one may ask *Given order n and weight k, does NCW(n,k) exist?*

Analytic Results

Letting $Y = NC(0100...0)$, then $Y^n = NC(-00...0) = -I_n$ and $Y^{2n} = NC(100...0) = I_n$. Then $NCW(n,k) = Y^{a_1} + Y^{a_2} + ... + Y^{a_k}$ where a_i is between 0 and 2n for all i, $a_i \neq a_j$ and $a_i \neq a_j + n$ for all i and j, $i \neq j$.

$$NCW(4,3) = Y^1 + Y^2 + Y^7$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 \\ 0 & - & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & - \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then, since $N = NCW(n,k)$ is a weighing matrix, $NN^t = kI$. And so, $NN^t = (Y^{a_1} + Y^{a_2} + ... + Y^{a_k})(Y^{2n-a_1} + Y^{2n-a_2} + ... + Y^{2n-a_k}) = kI$. By trying all possible values of a_i ($a_i \equiv -aj \pmod{2n}$) , $a_i \equiv a_j - a_k \pmod{2n}$, or $a_i - a_l \equiv a_j - a_k \pmod{2n}$), a proof by cases will tell us which values of $a_1, a_2, ..., a_k$ will produce such a matrix, as well as which values of n. Such an analysis was done to show that NCW(n,1) exists for all n (namely the Identity matrix), NCW(n,2) exists if and only if n is a multiple of 2, NCW(n,3) exists if and only if n is a multiple of 4, NCW(n,4) exists if and only if n is a multiple of 7, and NCW(n,5) exists if and only if n is a multiple of 6. Once a Negacyclic NCW(n,k) is found, NCW(mn,k) exists, we simply add m 0's between the entries of NCW(n,k).

$$NCW(2,1) \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$NCW(4,2) \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ - & 0 & 1 & 0 \\ 0 & - & 0 & 1 \end{bmatrix}$$

$$NCW(4,3) \quad \begin{bmatrix} 0 & 1 & 1 & - \\ 1 & 0 & 1 & 1 \\ - & 1 & 0 & 1 \\ - & - & 1 & 0 \end{bmatrix}$$

$$NCW(7,4) \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ - & 1 & 0 & 0 & 1 & 0 & 1 \\ - & - & 1 & 0 & 0 & 1 & 0 \\ 0 & - & - & 1 & 0 & 0 & 1 \\ - & 0 & - & - & 1 & 0 & 0 \\ 0 & - & 0 & - & - & 1 & 0 \\ 0 & 0 & - & 0 & - & - & 1 \end{bmatrix}$$

$$NCW(6,5) \quad \begin{bmatrix} 1 & 1 & 1 & - & 1 & 0 \\ 0 & 1 & 1 & 1 & - & 1 \\ - & 0 & 1 & 1 & 1 & - \\ 1 & - & 0 & 1 & 1 & 1 \\ - & 1 & - & 0 & 1 & 1 \\ - & - & 1 & - & 0 & 1 \end{bmatrix}$$

Computer Searches

A more direct way to answer our existence question is by computer searches. It can be shown that if NCW(n,k)=NCW($a_1 a_2 ... a_n$) exists, then the matrix A=circ($|a_1||a_2|...|a_n|$) has the follwing property: $AA^t = kI \pmod 2$. A Computer program can generate all sequences of the form ($b_1 b_2 ... b_n$) where k of the b_i's are 1. If circ($b_1 b_2 ... b_n$) is a boolean filter, then all possible sequences of the form ($c_1 c_2 ... c_n$) are generated, where $c_i = 0$ if $b_i = 0$, and $c_j = 1$ or $c_j = -1$ if $b_j = 1$. If NC($c_1 c_2 ... c_n$) is a weighing matrix, then NCW(n,k) has been found. If none of the matrices are weighinhg matrices, then one can conclude no NCW(n,k) exists.

Results Table

n/k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	Y																															
2	Y	Y																														
3	Y																															
4	Y	Y	Y																													
5	Y																															
6	Y	Y			Y																											
7	Y			Y																												
8	Y	Y	Y				Y																									
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27	Y																															
28	Y	Y	Y	Y				Y		Y		Y	Y					Y								Y	Y					
29	Y																															
30	Y	Y			Y				Y																Y				Y			
31	Y															Y									Y							
32	Y	Y	Y				Y																								Y	
33	Y																								Y							

- Green indicates an NCW exists
- Orange indicates we aren't sure
- White indicates we know there isn't one
- Dark Green represents an NCW not deriving directly from the Delsarte/Goethals/Seidel construction, or 'multiples' thereof

Applications

One might be asking *Given and k, does NCW(n,k) exist?* because of the applications of Negacyclic Weighing Matrices.

- NCW's are closely tied to Ryser's Conjecture, which states *No circulant Hadamard Matrix exists for orders greater than 4.* This connection is through Circulant Patrial Hadamard Matrices, which have applications in cryptology. Hadamard Matrices have applications in telecommunications, digital signals processing and quantum computing.
- NCW's are a subclass of Weighing Matrices. These matrices hace applications in statistics, engineering and cryptology.
- NCW(n,n-1) are a subclass of Conference Matrices. Conference Matrices wre first studied in telephony, they give instructions on how to construct Conference Networks (hence the name Conference Matrix).

Some Open Questions

1. *Does a Negacyclic weighing matrix of weight 6 exist?* From the results table, one can see that a NCW of weight 6 has yet to be found. It has been conjectured that no such matrix exists, but still not proven.
2. *Is there a better search algorithm?* The computer searches for NCW(n,k) for greater orders n may take several hours on a personal computer. Is there a better search algorithm that can produce results in less time?
3. *Is the analysis performed in the analytic results portion programmable?* The analysis performed will give definite results as to which orders n an NCW of weight k exist. But for large weights, there is a very large number of cases to go through. If the analysis is programmable, these cases could be evaluated with much greater speed.
4. *Given order n and weight k, does NCW(n,k) exist?* Of course, the ultimate goal is to answer this question. Perhaps there is a formula that will give a definite answer as to the existence of NCW(n,k) for any n and k, and a method to construct NCW(n,k) when it does exist.

Colin Desmarais

Patrick Naylor

Supervisor: **Robert Craigen**

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