# TESTING BI-ORDERABILITY OF KNOT GROUPS

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## WHAT IS A KNOT?

By a **knot**, we mean a mathematical representation of a knotted string, with the ends glued together. Figure 1 depicts the stevedore knot in these terms. Knots are studied for their complex topological properties. Applications of knot theory are found in string theory and problems related to protein folding and DNA unknotting.

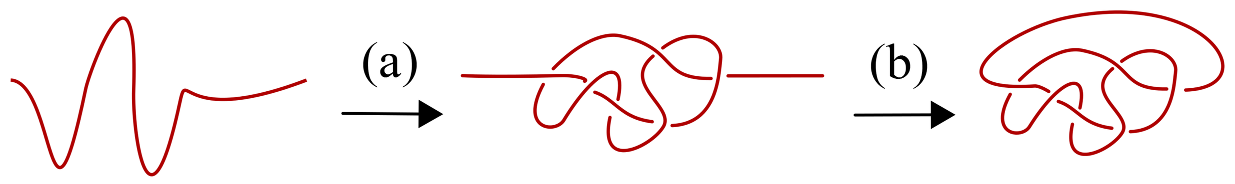


**Figure 1:** The stevedore knot. A knot is tied in (a), and the ends glued together in (b)

## BACKGROUND

Two knots are said to be equivalent if one can be changed to the other by continously deforming the space around them, via ambient isotopy (i.e. reshaped without cutting or gluing). Without much effort, it can be seen that the stevedore knot can be reshaped into $6_1$ in Figure 3. But can the same be done for $6_2$? The answer is no, although checking every single possible way to reshape a knot is not an easy feat. For this reason, we study properties that are invariant under such deformations. These properties are called **knot invariants**.

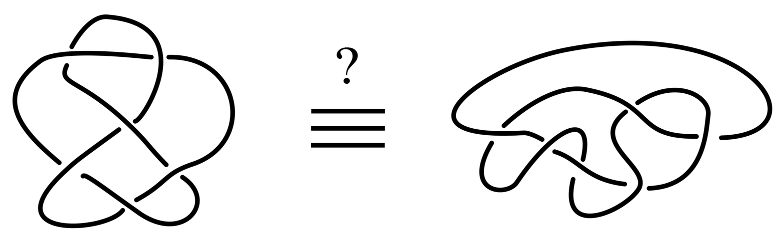


**Figure 2:** The knot $6_2$ on the left is not equivalent to the stevedore knot on the right

The **knot group** of a knot $K$, denoted $\pi(K)$, is a knot invariant and algebraic object that encodes information about the knot as well as its geometric properties. We can therefore study the algebraic properties of the knot group so as to study the knot itself. One such algebraic property is bi-orderability, meaning the elements of $\pi(K)$ can be given a strict total ordering that is invariant under mutliplication on the left or right by another element.

What may be surprising is knots can be tabulated in massive databases [1]. The knots are enumerated beginning with the simplest ones (fewer crossings), and the knots become more complicated (more crossings) as one reads through the table. Below is an example of a knot table listing all knots up to equivalence with 7 or fewer crossings.

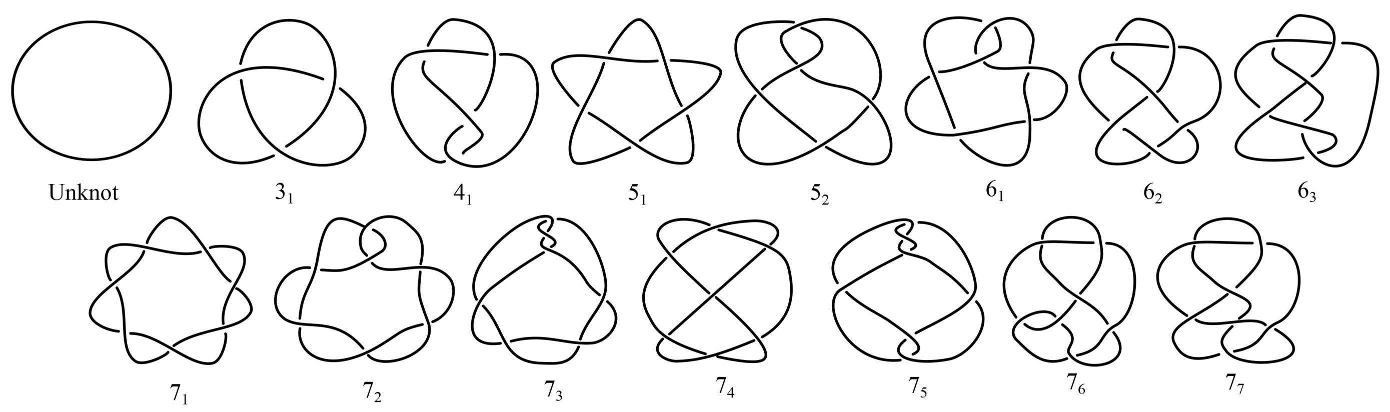


**Figure 3:** Knot table of all knots with 7 or fewer crossings

## OBJECTIVES

- **To discover new theorems that determine bi-orderability of knot groups and are simple to use.** Current theorems require complicated conditions on the topology of a knot $K$ or algebra of $\pi(K)$ in order to determine bi-orderability. We wish to provide tools to determine bi-orderability of knot groups based on simple properties of the knot.
- **To perform a computer survey of all knots with 12 or fewer crossings and provide a list of knots for which the bi-orderability of the knot group can be determined.** Previous theorems apply to 499 of all 2977 knots with 12 or fewer crossings (roughly 17%) [2]. Will the new theorems improve this number?

## METHODS

We will use all possible theorems to achieve our objectives, but specifically, we concentrate our attention on Theorem 1.

**Theorem 1.** [3] *Let $K$ be a knot, and suppose that $\pi(K)$ has a presentation of the form $\langle a, b | w\rangle$ where $w$ is tidy. Let $\Delta_K(t)$ denote the Alexander polynomial of $K$. Then:*

*1. If $\pi(K)$ is bi-orderable, then $\Delta_K(t)$ has a positive real root.*

*2. If $w$ is monic and all the roots of $\Delta_K(t)$ are real and positive, then $\pi(K)$ is bi-orderable.*

*3. If $w$ is principal, $\Delta_K(t) = a_0 + \cdots + a_{d-1}t^{d-1} - mt^d$ where $\gcd\{a_0, \ldots, a_{d-1}\} = 1$ and $a_{d-1}$ is not divisible by $m$, and all the roots of $\Delta_K(t)$ are real and positive, then $\pi(K)$ is bi-orderable.*

The Alexander polynomial $\Delta_K(t)$ is another easy-to-compute knot invariant. This new theorem can potentially be applied to many knots. Unfortunately, the conditions required on $\pi(K)$ can be quite tedious to verify. We wish to find infinite families of knots that satisfy these conditions, thus achieving the first objective. Next, we wish to explicitly find all knots with 12 or fewer crossings that satisfy the conditions, thus partly achieving the second objective.

## RESULTS

### THEORETICAL RESULTS

In line with the first objective, we were able to prove two major theorems involving 2-bridge knots. These knots are those that appear as in Figure 4, where each box $T_i$ represents twisting two segments of string together a certain number of times. These knots densily populate the knot table, with the first knot of bridge index greater than 2 being $8_5$, the $20^{th}$ knot in the table.

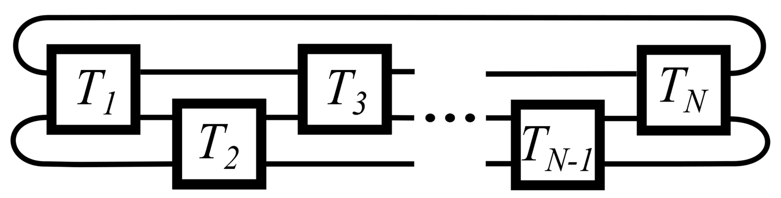


**Figure 4:** Standard construction of 2-bridge knots

This first theorem concerns all 2-bridge knots:

**Theorem 2.** *Suppose that $K$ is a two-bridge knot with Alexander polynomial $\Delta_K(t)$. If $\pi(K)$ is bi-orderable, then $\Delta_K(t)$ has a positive real root.*

The next theorem concerns twist knots. These make up an infinite subfamily of 2-bridge knots. Figure 5 depicts the twist knot $K_q$.

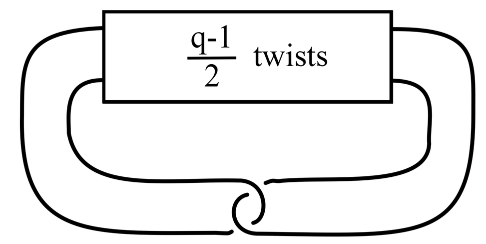


**Figure 5:** Twist knot $K_q$

With the following theorem we can determine bi-orderability of the knot groups of all twist knots.

**Theorem 3.** *Let $q > 3$ be an odd integer. If $q \equiv 1 \pmod 4$ then $\pi(K_q)$ is bi-orderable, and if $q \equiv 3 \pmod 4$ then $\pi(K_q)$ is not bi-orderable.*

### COMPUTATIONAL RESULTS

We wrote a computer that would take $\pi(K)$ as input for a given knot $K$, and output which conditions of Theorem 1 are satisfied. Next, with the aid of a computer, a survey of all 2977 knots with 12 or fewer crossings was completed: $\pi(K)$ was calculated using the computer program SnapPy [5], then Theorems 1, 2 and 3 were applied whenever possible, as well as the previous theorems in [2]. The following is a summary of the results concerning knots for which the bi-orderability of the knot group could not previously be determined:

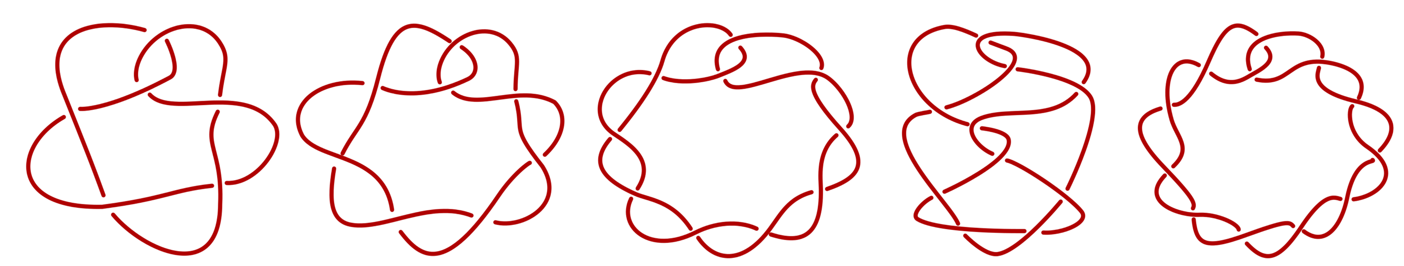

**Figure 6:** Newly discovered knots with bi-orderable knot groups. From left to right: $6_1, 8_1, 10_1, 10_{13}$ and $12a_{803}$

- 3 knots have bi-orderable knot groups by Theorem 3: $6_1, 8_1$ and $10_1$.
- 2 knots have bi-orderable knot groups by explicity verifying the conditions of Theorem 1: $10_{13}$ and $12a_{803}$.
- 79 knots have non-bi-orderable groups, by Theorem 2.
- 107 knots have non-bi-orderable groups by explicity verifying the conditions of Theorem 1.

Two mistakes in the literature [2] were rectified. Including previous results, the bi-orderability of 689 out of the first 2977 has been determined, which is roughly 23%.

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[1] J. C. Cha and C. Livingston. Kontinfo: Table of knot invariants. http://www.indiana.edu/∼knot-info (26/10/2014)
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