Constructions of hypergraphs free of certain 3-uniform 3-partite hypergraphs

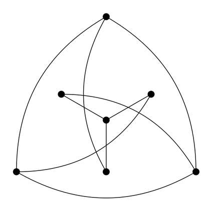
BACKGROUND

Definition 1. *Given a graph F, the extremal number for F, denoted $ex(n;F)$, is the maximum number of edges in a graph on n vertices that does not contain F as a weak subgraph.*

Turán[1] found the extremal numbers for all complete graphs in 1941, while for most graphs the exact extremal numbers are not known. In 1954, Kővári, Sós, and Turán[2] provided the following upper bound for bipartite graphs, where $s \le t$:

$$ex(n;K_{s,t}) \le \left(\frac{1}{2}(t-1)^{1/s} + o(1)\right) n^{2-1/s}.$$

Lower bounds for extremal numbers are achieved by providing constructions of F-free graphs with many edges. The constructions by Reiman[3] of $K_{2,2}$-free graphs using finite projective planes have $\left(\frac{1}{2}+o(1)\right)n^{3/2}$ edges.

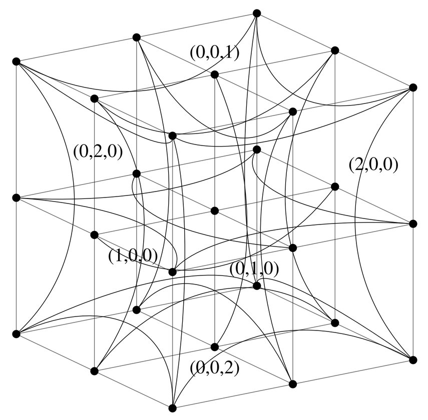


Construction of a $K_{2,2}$-free graph by Reimann using a coordinatization of the Fano Plane.

The exact extremal numbers for $K_{2,2}$ and specific values of n have been found, including the result by Füredi[4] that if $q \ge 13$ is a prime power and $n = q^2+q+1$, then $ex(n;K_{2,2}) = \frac{1}{2}q(q+1)^2$.
In 1966, Brown[5] provided the following construction of $K_{3,3}$-free graphs:

Definition 2. *Fix an odd prime p. Let $r \neq 0$ be a quadratic residue modulo p if $p \equiv 3 \pmod 4$, and a quadratic nonresidue if $p \equiv 1 \pmod 4$. Define the graph $G = (V,E)$ as follows:*

- *Let $V = \mathbb{F}_p^3$; then G has p^3 vertices.*
- *For all $(a,b,c) \in \mathbb{F}_p^3$, define the sphere*
$$S(a,b,c) = \left\{(x,y,z) \in \mathbb{F}_p^3 \mid (x-a)^2+(y-b)^2+(z-c)^2 \equiv r \pmod p\right\}.$$
These spheres have p^2-p elements.
- *$\{\boldsymbol{u},\boldsymbol{v}\} \in E$ if and only if $\boldsymbol{u} \in S(\boldsymbol{v})$; there are $\frac{p^3}{2}(p^2-p)$ edges.*



Brown's construction for $p=3$.

Similar to spheres in $\mathbb{R}^3$, three spheres in $\mathbb{F}_p^3$ intersect in at most 2 points, and so there is no $K_{3,3}$. Letting $n = p^3$, G has $\frac{n^{5/3}-n^{4/3}}{2}$ edges, and so

$$ex(n;K_{3,3}) \ge \left(\frac{1}{2}+o(1)\right)n^{5/3}.$$

Füredi[6] showed that $ex(n;K_{3,3}) \le \left(\frac{1}{2}+o(1)\right)n^{5/3}$, making the bound tight. For a prime power q, Alon, Rónyai, and Szabó[7] provided a construction with $n = q^3-q^2$ vertices and $\frac{1}{2}(q^5-q^4-q^3+q^2)$ edges, and so improve the $o(1)$ factor in the lower bound for $ex(n;K_{3,3})$. In fact, they show that for $t \ge (s-1)!+1$, $ex(n;K_{s,t}) \ge \left(\frac{1}{2}+o(1)\right)n^{2-1/s}$.

1 P. Turán, Eine Extremalaufgave aus der Graphentheorie, *Math. Fiz Lapook* **48** (1941), 436-453.
2 T. Kővári, V. T. Sós, and P. Turán, On a problem of K. Zarankiewicz, *Colloq. Math.* **3** (1954), 50-57.
3 I. Reimann, Uber ein Problem von K. Zarankiewicz, *Acta Math. Acad. Sci. Hung.* **9** (1959), 269-279.
4 Z. Füredi, On the number of edges of quadrilateral-free graphs *J. Comb. Theory Ser. B* **68** (1996), 1-6.
5 W. G. Brown, On graphs that do not contain a Thomsen graph, *Canad. Math. Bull.* **8** (1966), 281-285.
6 Z. Füredi, An upper bound on Zarankiexicz' problem, *Combin., Probab. Comput.* **5** (1996), 29-33.
7 N. Alon, L. Rónyai, and T. Szabó, Norm-graphs: variations and applications, *J. Combin. Theory Ser. B* **76** (1999), 280-290.

FORBIDDING $K^{(3)}(2,2,2)$

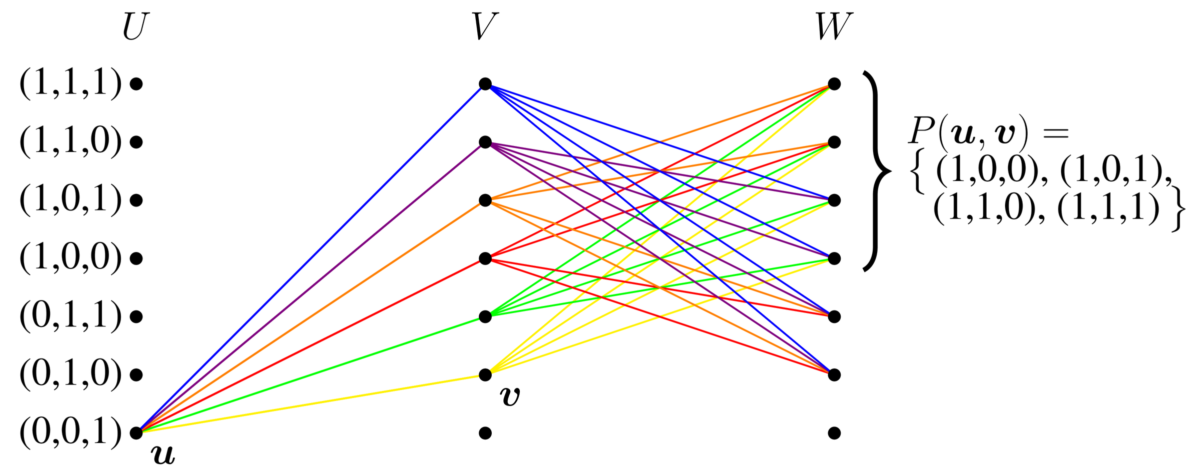
The definition of extremal numbers generalizes to hypergraphs, where even less is known. For the d-uniform d-partite hypergraphs $K^{(d)}(2,2,...,2)$, Erdős[8] showed that there exists a constant c_d for which $ex(n;K^{(d)}(2,2,...,2)) \le c_d n^{d-\frac{1}{2^{d-1}}}$.
In 1999, Gunderson, Rödl, and Sidorenko[9] used a probabilisitc argument to show that

$$ex(n;K^{(3)}(2,2,2)) \ge (1+o(1))\frac{n^{13/5}}{4\cdot 3^{8/5}}.$$

Their argument is extended to $K^{(d)}(2,2,...,2)$, where it achieves the best known lower bound when $d \ge 4$. However, the bound for $ex(n;K^{(3)}(2,2,2))$ can be improved with the construction below:

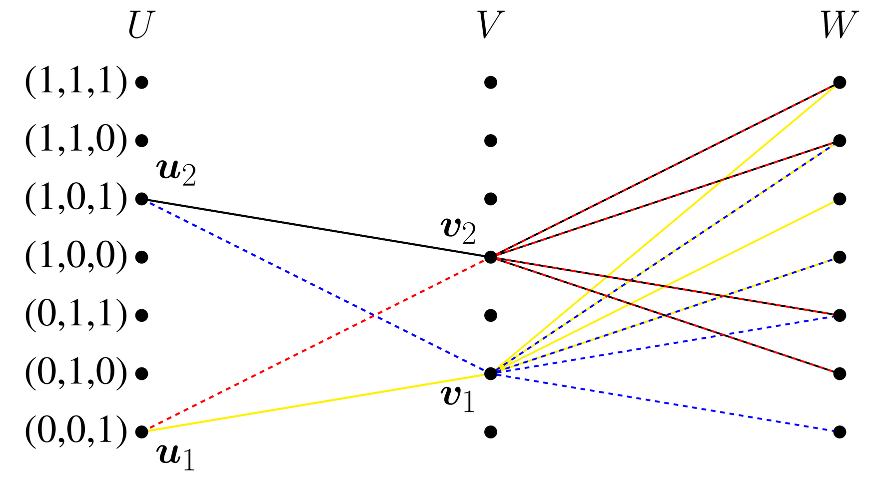
Definition 3 (Desmarais, D. Gunderson)**.** *For a prime power q, define a 3-uniform 3-partitie hypergraph $\mathcal{H} = (U \cup V \cup W, \mathcal{E})$ as follows:*

- *Let $U = V = W = \mathbb{F}_q^3 \backslash \{(0,0,0)\}$. $\mathcal{H}$ has $3(q^3-1)$ vertices.*
- *For $\boldsymbol{u},\boldsymbol{v} \in \mathbb{F}_q^3$, define the plane $P(\boldsymbol{u},\boldsymbol{v}) = \left\{\boldsymbol{w} \in \mathbb{F}_q^3 \mid \boldsymbol{w} \bullet (\boldsymbol{u} \times \boldsymbol{v}) = 1\right\}$.*
- *$\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\} \in \mathcal{E}$ if and only if $\boldsymbol{w} \in P(\boldsymbol{u},\boldsymbol{v})$. $\mathcal{H}$ has $(q^3-1)(q^3-q)q^2$ hyperedges.*



Construction for $q=2$, with all hyperedges containing $(0,0,1) \in U$ shown.

The hypergraph $\mathcal{H}$ is $K^{(3)}(2,2,2)$-free since for any $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1$ and $\boldsymbol{v}_2$, the four planes $P(\boldsymbol{u}_1,\boldsymbol{v}_1)$, $P(\boldsymbol{u}_1,\boldsymbol{v}_2)$, $P(\boldsymbol{u}_2,\boldsymbol{v}_1)$ and $P(\boldsymbol{u}_2,\boldsymbol{v}_2)$ intersect in at most 1 element.



The planes $P(\boldsymbol{u}_1,\boldsymbol{v}_1)$ in yellow, $P(\boldsymbol{u}_1,\boldsymbol{v}_2)$ in red, $P(\boldsymbol{u}_2,\boldsymbol{v}_1)$ in blue, and $P(\boldsymbol{u}_2,\boldsymbol{v}_2)$ in black.

Letting $n = 3(q^3-1)$, the construction above provides a bound of

$$ex(n;K^{(3)}(2,2,2)) \ge (1+o(1))\left(\frac{n}{3}\right)^{8/3}.$$

A construction similar to the one above achieving the same bound in the exponent was described by Katz, Kropp, and Maggioni in 2002[10], and another construction presented by Cilleruelo and Tesoro[11] also has the same bound.

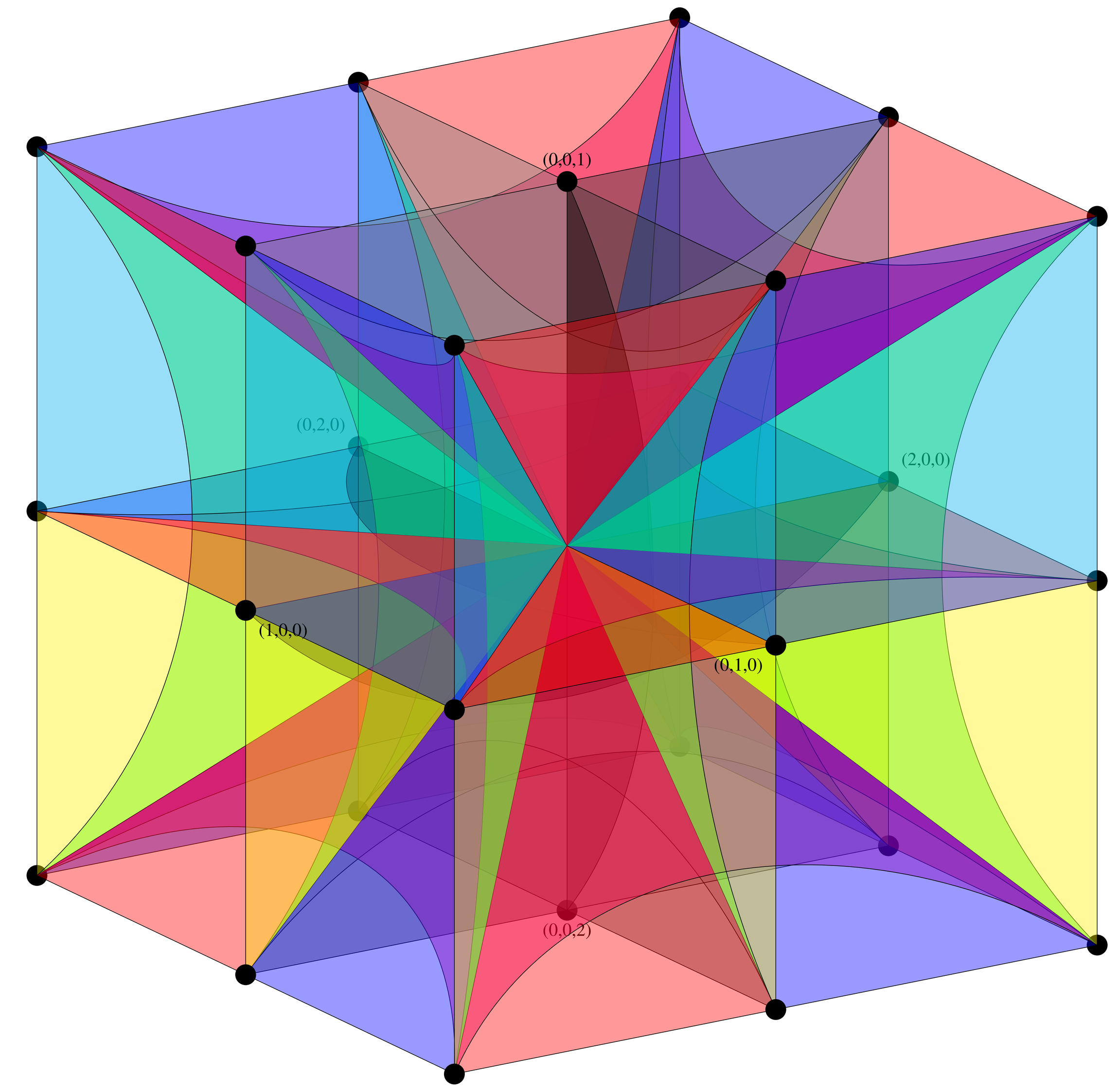
8 P. Erdős, On extremal problems of graphs and generalized graphs, *Israel J. Math.* **2** (1964), 183-190.
9 D. S. Gunderson, V. Rödl, and A. Sidorenko, Extremal problems for sets forming Boolean algebras and complete partite hypergraphs, *J. Combin. Theory Ser. A.* **88** (1999), 342-367.
10 N. H. Katz, E. Krop, and M. Maggioni, Remarks on the box problem *Math. Res. Lett.* **9** (2002), 515-519.
11 J. Cilleruelo and R. Tesoro, On sets free of sumsets with summands of prescribed size, Preprint arXiv:1504.00137, (2015).

FORBIDDING $K^{(3)}(2,2,3)$

Since $K^{(3)}(2,2,2)$ is a subgraph of $K^{(3)}(2,2,3)$, any lower bound for $ex(n;K^{(3)}(2,2,2))$ is also a bound for $ex(n;K^{(3)}(2,2,3))$. However, the coefficient in the bound can be improved with the following construction:

Definition 4 (Desmarais)**.** *Fix an odd prime p. Let $r \neq 0$ be a quadratic residue modulo p if $p \equiv 3 \pmod 4$, and a quadratic nonresidue if $p \equiv 1 \pmod 4$. Define the 3-uniform hypergraph $\mathcal{G} = (V,\mathcal{E})$ as follows:*

- *Let $V = \mathbb{F}_p^3 \backslash \{(0,0,0)\}$; then $\mathcal{G}$ has p^3 vertices.*
- *For all $(a,b,c) \in \mathbb{F}_p^3$, define the sphere*
$$S(a,b,c) = \left\{(x,y,z) \in \mathbb{F}_p^3 \mid (x-a)^2+(y-b)^2+(z-c)^2 \equiv r \pmod p\right\}.$$
These spheres have p^2-p elements.
- *$\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\} \in \mathcal{E}$ if and only if $\boldsymbol{w} \in S(-\boldsymbol{v}-\boldsymbol{u})$; there are $\frac{1}{3}\binom{p^3}{2}(p^2-p)$ hyperedges.*



Construction for $p=3$, with all hyperedges containing $(0,0,0) \in V$ shown.

With $n = p^3$, the construction gives a lower bound of

$$ex(n;K^{(3)}(2,2,3)) \ge \left(\frac{1}{6}+o(1)\right)n^{8/3}.$$

A construction by Mubayi 2002[12] that extends the $K_{3,3}$-free constructions of Alon, Rónyai, and Szabó[7] achieves this same bound, but improves on the $o(1)$ factor.

12 D. Mubayi, Some exact results and new asymptotics for hypergraph Túran numbers, *Combin., Prob. Comp.* **11** *(2002), 299-309.*

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